

PARTITION FUNCTIONS, MAPPING CLASS GROUPS AND DRINFELD DOUBLES

Jens Fjelstad ^a, Jürgen Fuchs ^b, Christoph Schweigert ^c, Carl Stigner ^b

^a *Department of Physics, Nanjing University
22 Hankou Road, Nanjing, 210093 China*

^c *Teoretisk fysik, Karlstads Universitet
Universitetsgatan 21, S-651 88 Karlstad*

^c *Organisationseinheit Mathematik, Universität Hamburg
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D-20 146 Hamburg*

ABSTRACT

Higher genus partition functions of two-dimensional conformal field theories have to be invariants under linear actions of mapping class groups. We illustrate recent results [4, 6] on the construction of such invariants by concrete expressions obtained for the case of Drinfeld doubles of finite groups. The results for doubles are independent of the characteristic of the underlying field, and the general results do not require any assumptions of semisimplicity.

1 General results

To start, let us sketch the incentive for our work from two-dimensional conformal field theory (CFT). The building blocks for CFT correlators with given chiral data are the conformal blocks, which form vector bundles with projectively flat connection over the moduli space of complex curves with marked points. It is fairly well understood that essential properties of the monodromies of these connections are encoded in the structure of a ribbon category \mathcal{C} . In many applications to physical systems, it is unnatural to require \mathcal{C} to be semisimple. Indeed, non-semisimple ribbon categories arise in logarithmic conformal field theories, a topic of much recent activity and with many applications, in particular to statistical systems.

In this note we consider linear representations of the mapping class groups of Riemann surfaces of genus g with m holes, for arbitrary values of g and m , on complex vector spaces that are specific morphism spaces of a finite ribbon category \mathcal{C} . Recall that a *finite tensor category* is a rigid monoidal category with finite-dimensional morphism spaces and finitely many isomorphism classes of simple objects, each of which has a projective cover, and with every object having finite length; a *ribbon category* is a rigid monoidal category endowed with a compatible braiding and twist.

A central idea in quantum field theory is the one of “summing over all intermediate states”. Beyond the semisimple setting, one cannot give a naive mathematical meaning to this by restricting the summation by hand to, say, simple or indecomposable or projective objects. The idea is, however, appropriately implemented by the categorical notion of a *coend* $\int^{X \in \mathcal{C}} G(X, X)$ of a certain functor $G: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$. The coend can be seen as a formalization of the idea to sum over all possible states, in a way that is consistent with the morphisms in the category. We have [9, 8]

Theorem 1 *Let \mathcal{C} be a finite ribbon category.*

(i) *The coend*

$$L(\mathcal{C}) := \int^{U \in \mathcal{C}} U^\vee \otimes U \quad (1)$$

of the functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ that acts on objects as $(U, V) \mapsto U^\vee \otimes V$ exists.

(ii) *The coend $L(\mathcal{C})$ carries a natural structure of a Hopf algebra in \mathcal{C} . It is endowed with an integral and with a Hopf pairing $\varpi_{L(\mathcal{C})}: L(\mathcal{C}) \otimes L(\mathcal{C}) \rightarrow \mathbf{1}$.*

(iii) *Via the rigidity of \mathcal{C} , every object of \mathcal{C} carries a natural structure of right comodule over $L(\mathcal{C})$.*

Proposition 2 [6, Rem. 2.3] *For \mathcal{C} a finite ribbon category, every object of \mathcal{C} carries a natural structure of left-right Yetter-Drinfeld module over $L(\mathcal{C})$, with the right comodule structure being the one of Theorem 1(iii) and the left module structure obtained via the braiding of \mathcal{C} .*

This provides a fully faithful embedding of \mathcal{C} into the category of left-right Yetter-Drinfeld modules over $L(\mathcal{C})$.

A *factorizable* finite ribbon category is a finite ribbon category \mathcal{C} for which the Hopf pairing $\varpi_{L(\mathcal{C})}$ is non-degenerate; for such a category the integral of the Hopf algebra $L(\mathcal{C})$ is two-sided and $L(\mathcal{C})$ also has a two-sided cointegral (see e.g. Proposition 5.2.10 and Corollary 5.2.11 of [8]). In the sequel we restrict our attention to a specific subclass of such categories. We denote by \mathbb{k} an algebraically closed field of characteristic zero and by $H \equiv (H, m, \eta, \Delta, \varepsilon, S, R, \nu)$ a finite-dimensional ribbon Hopf \mathbb{k} -algebra. Here $m, \eta, \Delta, \varepsilon$ and S are the product, unit, coproduct,

counit and antipode of H , $R \in H \otimes_{\mathbb{k}} H$ is the R -matrix and $v \in H$ the ribbon element of H . Such a Hopf algebra has, uniquely up to scalars, a non-zero left integral $\Lambda \in H$ and a right cointegral $\lambda \in H^*$; the antipode of H is invertible. We denote by $Q := R_{21} \cdot R \in H \otimes_{\mathbb{k}} H$ the monodromy matrix and by $f_Q = (d_H \otimes id_H) \circ (id_{H^*} \otimes Q): H^* \rightarrow H$ the *Drinfeld map*.

We also assume that the Hopf algebra H is *factorizable*, meaning that the Drinfeld map f_Q is invertible. In this case the integral Λ is two-sided (in other words, the Hopf algebra H is *unimodular*), and the Drinfeld map sends the cointegral λ to a non-zero multiple of the integral. We can (and do) normalize them such that $\lambda \circ \Lambda = 1$ and $f_Q(\lambda) = \Lambda$.

Complex Hopf algebras with structure very close to the one considered here arise in the description of representation categories of chiral algebras that are not semisimple, see e.g. [2, 3, 11, 13, 12].

Now denote by $H\text{-Mod}$ the category of (finite-dimensional, left) H -modules. The following is an immediate consequence of well-known results:

Proposition 3 *The category $H\text{-Mod}$ carries a natural structure of a factorizable finite ribbon category. Specifically, the tensor product of $H\text{-Mod}$ is obtained by pull-back of the coproduct Δ of H , the left and right dualities come from the antipode of H and its inverse, the braiding comes from the R -matrix R , and the twist comes from the ribbon element v .*

The category $H\text{-Bimod}$ of finite-dimensional H -bimodules can be treated very much in the same vein, giving (see [4])

Corollary 4 *The category $H\text{-Bimod}$ of finite-dimensional H -bimodules has a natural structure of a factorizable finite ribbon category.*

Remark 5 (i) *It is worth pointing out that the relevant monoidal structure is not the one for which the tensor product is taken over H (and thus treats the Hopf algebra H just as an algebra). Braidings for that other monoidal structure are discussed in [1].*

(ii) *We can use the ribbon structure on H to equip the category $H\text{-Bimod}$ (endowed with the relevant monoidal structure) with a natural structure of a ribbon category. To this end, we can use the simultaneous left action and right action of either the R -matrix and the ribbon element, or else the inverse R -matrix and inverse ribbon element. Altogether this results in four structures of ribbon category on $H\text{-Bimod}$; these are pairwise isomorphic. For our purpose it is crucial to take one of the two ribbon structures in which mutually inverse elements are used on the left and on the right; for concrete expressions see the formulas (3.3) and (4.19) of [4]. This is precisely the ribbon structure that makes $H\text{-Bimod}$, for factorizable H , braided equivalent to the Drinfeld center of $H\text{-Mod}$.*

As shown in Appendix A.2 of [4], there is an equivalence

$$H\text{-Mod}^{\text{rev}} \boxtimes H\text{-Mod} \xrightarrow{\simeq} H\text{-Bimod} \quad (2)$$

of ribbon categories, where \boxtimes is the Deligne tensor product of abelian categories and the *reverse* category \mathcal{C}^{rev} of a ribbon category \mathcal{C} is obtained from \mathcal{C} by inverting the braiding and the twist isomorphisms. We use this equivalence to tacitly identify these two categories. In particular, we think of the coend $L(H\text{-Mod}^{\text{rev}} \boxtimes H\text{-Mod})$, as introduced in (1), as an H -bimodule. We

denote this bimodule by K ; its structure as an H -bimodule and as a Hopf algebra in $H\text{-Bimod}$ are described in detail in Appendix A.3 of [4].

The simplest partition function in the semisimple case is the charge conjugation modular invariant; this is of Peter-Weyl form and thus a direct sum of terms in bijection with the isomorphism classes of simple H -modules. In the non-semisimple case, the direct sum gets again generalized to a coend, this time of a functor $G^H : H\text{-Mod}^{\text{op}} \times H\text{-Mod} \rightarrow H\text{-Bimod}$. The functor G^H in question is the one obtained by composing the functor $H\text{-Mod}^{\text{op}} \times H\text{-Mod} \rightarrow H\text{-Mod}^{\text{rev}} \boxtimes H\text{-Mod}$ that acts on objects as $U \times V \mapsto U^\vee \boxtimes V$ with the equivalence (2). The coend of G^H implements in a consistent manner the idea to pair left and right movers in the form of charge conjugation on *all* representations.

Proposition 6 [4, Propos. A.3] *The coend F of the functor $G^H : H\text{-Mod}^{\text{op}} \times H\text{-Mod} \rightarrow H\text{-Bimod}$ exists. Its underlying object – also denoted by F – is the coregular bimodule, i.e. the vector space H^* dual to H endowed with the duals of the left and right regular actions of H on itself.*

Similarly as the coend K , also the coend F turns out to carry important additional algebraic structure internal to $H\text{-Bimod}$:

Theorem 7 [5, Theorem 2] *The coend F carries a natural structure of a commutative symmetric Frobenius algebra with trivial twist in the ribbon category $H\text{-Bimod}$. The algebra and coalgebra structural maps are given by*

$$\begin{aligned} m_F &:= \Delta^*, & \eta_F &:= \varepsilon^*, & \varepsilon_F &:= \Lambda^* & \text{and} \\ \Delta_F &:= [(id_H \otimes (\lambda \circ m)) \circ (id_H \otimes s \otimes id_H) \circ (\Delta \otimes id_H)]^*, \end{aligned} \tag{3}$$

with Λ and λ the integral and cointegral of H , respectively.

We now turn to the action of mapping class groups. The results in [9] imply that for any triple of non-negative integers g , p and q , the morphism space $V_{g;p,q} = \text{Hom}_{H|H}(K^{\otimes g} \otimes F^{\otimes q}, F^{\otimes p})$ naturally carries a projective representation $\pi_{g;p,q}$ of the subgroup $\text{Map}_{g;p,q}$ of the mapping class group of Riemann surfaces of genus g with $p+q$ holes that leaves two selected subsets of sizes p and q separately invariant. For details see Propos. 2.4 and Remark 2.6 of [6]. $\pi_{g;p,q}$ is in fact a genuine linear representation, see Remark 5.5 of [4].

In applications to CFT, F is a candidate for the bulk state space, and the spaces $V_{g;p,q}$ play the role of genus- g conformal blocks with p incoming and q outgoing insertions of the bulk state space. The corresponding correlation functions are specific elements in those spaces that have to be invariant under the mapping class group action and be compatible with sewing.

We now present elements $\text{Cor}_{g;p,q} \in V_{g;p,q}$ that are candidates for these correlation functions. Denote by $m_F^{(r)}$ and $\Delta_F^{(r)}$ multiple products and coproducts of F , respectively, and by ρ_F^K the natural action (see Propos. 2) of the Hopf algebra $K \in H\text{-Bimod}$ on the H -bimodule F , and set

$$\begin{aligned} \text{Cor}_{0;1,1} &:= id_F, & \text{Cor}_{1;1,1} &:= m_F \circ (\rho_F^K \otimes id_F) \circ (id_K \otimes \Delta_F), \\ \text{Cor}_{g;1,1} &:= \text{Cor}_{1;1,1} \circ (id_K \otimes \text{Cor}_{g-1;1,1}) \quad \text{for } g > 1, \\ \text{Cor}_{g;p,q} &:= \Delta_F^{(p)} \circ \text{Cor}_{g;1,1} \circ (id_{K^{\otimes g}} \otimes m_F^{(q)}). \end{aligned} \tag{4}$$

Theorem 8 *The element $\text{Cor}_{g;p,q}$ of $V_{g;p,q}$ is invariant under the action $\pi_{g;p,q}$ of the group $\text{Map}_{g;p,q}$.*

This statement is the main result of [6]. It provides yet another instance of a surprising conspiracy of algebraic structures – in the case at hand, finite-dimensional factorizable Hopf algebras – and geometric ones – here, the fundamental groups of moduli spaces of complex curves. We find it remarkable that semisimplicity of the categories involved is needed neither for producing mapping class group representations nor for the construction of physically motivated invariants.

2 Drinfeld doubles

Associated to any finite group G there is the quasitriangular Hopf algebra $\mathcal{D}G$, called the (Drinfeld) double of G . As a vector space, $\mathcal{D}G = \mathbb{k}(G) \otimes_{\mathbb{k}} \mathbb{k}[G]$ with $\mathbb{k}(G)$ the algebra of functions on G and $\mathbb{k}[G]$ the group algebra over an (algebraically closed) field \mathbb{k} . From here on we do not assume any longer that \mathbb{k} has characteristic zero, and thus do not require $\mathcal{D}G$ to be semisimple. A natural basis of $\mathcal{D}G$ is given by $b_{g|x} := \delta_g \otimes b_x$ with $g, x \in G$, where δ_g and b_x are the usual natural bases of $\mathbb{k}(G)$ and $\mathbb{k}[G]$. In terms of this basis the Hopf algebra structure of $\mathcal{D}G$ reads

$$\begin{aligned} b_{g|x} \cdot b_{h|y} &= \delta_{g, xhx^{-1}} b_{g|xy}, \quad \eta(1) = b_{1|e} = \sum_{g \in G} b_{g|e}, \quad \varepsilon(b_{g|x}) = \delta_{g,e}, \\ \Delta(b_{g|x}) &= \sum_{h \in G} b_{h|x} \otimes b_{h^{-1}g|x}, \quad S(b_{g|x}) = b_{x^{-1}g^{-1}x|x^{-1}}. \end{aligned} \quad (5)$$

Note that $S^2 = id_{\mathcal{D}G}$. The Hopf algebra $\mathcal{D}G$ is factorizable quasitriangular, with R-matrix and associated monodromy matrix given by

$$R = \sum_{g \in G} b_{g|e} \otimes b_{1|g} \quad \text{and} \quad Q = \sum_{g, h \in G} b_{h|g} \otimes b_{g|g^{-1}hg}. \quad (6)$$

$\mathcal{D}G$ is also ribbon, with ribbon element $\nu = \sum_{g \in G} b_{g|g^{-1}}$, and it has a two-sided integral and a two-sided cointegral, given (upon choice of normalizations) by $\Lambda = \sum_g b_{e|g}$ and by $\lambda = \sum_g \beta_{g|e} \in \mathcal{D}G^*$, where $\{\beta_{g|x}\}$ is the basis of the dual space $\mathcal{D}G^*$ dual to $\{b_{g|x}\}$. The integral satisfies $\varepsilon \circ \Lambda = |G|$; accordingly, by Maschke's theorem the category $\mathcal{D}G\text{-Mod}$ of left $\mathcal{D}G$ -modules is semisimple iff the characteristic of \mathbb{k} does not divide the order $|G|$.

3 Invariants from Drinfeld doubles

Let us apply some of the general results from Section 1 to the factorizable Hopf algebra $H = \mathcal{D}G$. First, the Hopf algebra K in $\mathcal{D}G\text{-Bimod}$ is $(\mathcal{D}G^*) \otimes_{\mathbb{k}} (\mathcal{D}G^*)$ as a vector space, with the coadjoint left and right $\mathcal{D}G$ -actions on the first and second tensor factor, respectively, and its Hopf algebra structure is easily expressed in terms of the dual basis $\{\beta_{g|x}\}$. The Hopf algebra structure of $K \in \mathcal{D}G\text{-Bimod}$, expressed in terms of the dual basis $\{\beta_{g|x}\}$, is given by

$$\begin{aligned} (\beta_{g|x} \otimes \beta_{h|y}) \cdot (\beta_{p|u} \otimes \beta_{q|v}) &= \delta_{px, up} \delta_{qvy, vqv} \beta_{pg|u} \otimes \beta_{v^{-1}qvhv^{-1}q^{-1}vq|v}, \\ \Delta_K(\beta_{g|x} \otimes \beta_{h|y}) &= \sum_{u, v \in G} \beta_{g|v} \otimes \beta_{u^{-1}hu|u^{-1}y} \otimes \beta_{v^{-1}gv|v^{-1}x} \otimes \beta_{h|u}, \\ 1_K &= \sum_{x, y \in G} \beta_{e|x} \otimes \beta_{e|y}, \quad \varepsilon_K(\beta_{g|x} \otimes \beta_{h|y}) = \delta_{x,e} \delta_{y,e}, \\ S_K(\beta_{g|x} \otimes \beta_{h|y}) &= \beta_{x^{-1}g^{-1}x|x^{-1}g^{-1}x^{-1}gx} \otimes \beta_{h^{-1}y^{-1}h^{-1}yh|h^{-1}y^{-1}h}. \end{aligned} \quad (7)$$

The Frobenius algebra F in $\mathcal{DG}\text{-Bimod}$ is \mathcal{DG}^* as a vector space, with the coregular left and right \mathcal{DG} -actions. The Frobenius algebra structure is obtained by specializing formula (3) to the present situation; we find

$$\begin{aligned}\eta_F &= \sum_{x \in G} \beta_{e|x}, & m_F(\beta_{g|x}, \beta_{h|y}) &= \delta_{x,y} \beta_{hg|x}, \\ \varepsilon_F(\beta_{g|x}) &= \delta_{g,e}, & \Delta_F(\beta_{g|x}) &= \sum_{h \in G} \beta_{h^{-1}g|x} \otimes \beta_{h|x}.\end{aligned}\tag{8}$$

Now consider the action ρ_F^K of the Hopf algebra K on the Frobenius algebra F , which is a linear map from $(\mathcal{DG}^*)^{\otimes_k 3}$ to \mathcal{DG}^* . We obtain

$$\rho_F^K(\beta_{g|x} \otimes \beta_{h|y} \otimes \beta_{k|z}) = \delta_{z,xzy} \delta_{k,x} \beta_{g^{-1}xg|g^{-1}zy^{-1}hy}.\tag{9}$$

We can now insert the specific expressions (8) and (9) into the general formula (4). We restrict our attention to the case $p = 1 = q$; the extension to general values of p and q is straightforward. At genus 1 we have

$$\text{Cor}_{1;1,1}(\beta_{g|x} \otimes \beta_{h|y} \otimes \beta_{k|z}) = \delta_{y^{-1}, z^{-1}xz} \delta_{y^{-1}hy, z^{-1}gz} \beta_{k[x,g]|z},\tag{10}$$

with $[x, g] = x^{-1}g^{-1}xg$ the group commutator. By iteration we arrive at

$$\text{Cor}_{n;1,1}(\beta_{gh|xy}^{(n)} \otimes \beta_{k|z}) = \prod_{i=1}^n \delta_{y_i^{-1}, z^{-1}x_i z} \delta_{y_i^{-1}h_i y_i, z^{-1}g_i z} \beta_{k[x_n, g_n] \cdots [x_1, g_1]|z},\tag{11}$$

where we introduced the short-hand notation

$$\beta_{gh|xy}^{(n)} := \beta_{g_1|x_1} \otimes \beta_{h_1|y_1} \otimes \beta_{g_2|x_2} \otimes \beta_{h_2|y_2} \otimes \cdots \otimes \beta_{g_n|x_n} \otimes \beta_{h_n|y_n}.\tag{12}$$

According to Theorem 8 the morphisms (12) are invariant under the action of $\text{Map}_{g;p,q}$ if \mathbb{k} has characteristic zero. But we actually expect that this remains true for general characteristic, including the case that the characteristic divides the order of G so that \mathcal{DG} is non-semisimple.

Indeed, we have verified that the torus partition function $\text{Cor}_{1;0,0}$ is modular invariant irrespective of the characteristic of \mathbb{k} . The action of the S- and T-transformation on $\text{Cor}_{1;0,0}$ is given by precomposition with the morphisms S_K and T_K depicted in (4.3) of [6], which in the case at hand read

$$T_K(\beta_{g|x} \otimes \beta_{h|y}) = \beta_{g|gx} \otimes \beta_{h|h^{-1}y}\tag{13}$$

and

$$S_K(\beta_{g|x} \otimes \beta_{h|y}) = \beta_{g^{-1}xg|g^{-1}} \otimes \beta_{y^{-1}|y^{-1}hy}.\tag{14}$$

Checking mapping class group invariance of correlators with field insertions and at higher genus amounts to specializing various expressions from [6, Sect. 5]; this would be straightforward, but some of the computations involved tend to be lengthy.

ACKNOWLEDGMENTS: This work is supported in part by the ESF Research Networking Programme “Interactions of Low-Dimensional Topology and Geometry with Mathematical Physics”. The collaboration between JFj, JFu and CSt is also supported by the VR Swedish Research Links Programme under project no. 348-2008-6049; JFu is in addition supported by VR under project no. 621-2009-3993. JFj is supported by 985 Project Grants from the Chinese Ministry of Education, by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), and by the China Science Postdoc grant 2011M500887. CSc is partially supported by the Collaborative Research Centre 676 “Particles, Strings and the Early Universe - the Structure of Matter and Space-Time” and by the DFG Priority Programme 1388 “Representation Theory”.

References

- [1] A.L. Agore, S. Caenepeel, and G. Militaru, e-print math.QA/1108.2575.
- [2] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, Commun. Math. Phys. 265 (2006) 47.
- [3] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, J. Math. Phys. 48 (2007) 032303.
- [4] J. Fuchs, C. Schweigert, and C. Stigner, J. of Algebra 363 (2012) 29.
- [5] J. Fuchs, C. Schweigert, and C. Stigner, in: *Strings, Gauge Fields, and the Geometry Behind. The Legacy of Maximilian Kreuzer* (A. Rebhan et al, eds.; World Scientific, Singapore 2012), p. 289.
- [6] J. Fuchs, C. Schweigert, and C. Stigner, e-print math.QA/1207.6863.
- [7] T. Kerler, in: *Quantum Invariants and Low-Dimensional Topology* (J.E. Andersen et al., eds; Dekker, New York 1997), p. 503.
- [8] T. Kerler and V.V. Lyubashenko, *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners* (Springer Verlag, New York 2001).
- [9] V.V. Lyubashenko, Commun. Math. Phys. 172 (1995) 467.
- [10] V.V. Lyubashenko, J. of Pure and Applied Algebra 98 (1995) 279.
- [11] K. Nagatomo and A. Tsuchiya, in: *Exploring New Structures and Natural Constructions in Mathematical Physics* (K. Hasegawa et al., eds.; Math. Soc. of Japan, Tokyo 2010), p. 1.
- [12] I. Runkel, M.R. Gaberdiel, and S. Wood, e-print 1201.6273.
- [13] A. Tsuchiya and S. Wood, e-print 1201.0419.